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Minimal projections in spaces of functions of N variables

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Abstract

We will construct a minimal and co-minimal projection from $L^p([0,1]^n)$ onto $L^p([0,1]^{n_1}) + \cdots + L^p([0,1]^{n_k})$, where $n = n_1 + \cdots + n_k$ (see Theorem 2.9). This is a generalization of a result of Cheney, Halton and Light from (Approximation Theory in Tensor Product Spaces, Lecture Notes in Mathematics, Springer, Berlin, 1985; Math. Proc. Cambridge Philos. Soc. 97 (1985) 127; Math. Z. 191 (1986) 633) where they proved the minimality in the case $n = 2$. We provide also some further generalizations (see Theorems 2.10 and 2.11 (Orlicz spaces) and Theorem 2.8). Also a discrete case (Theorem 2.2) is considered. Our approach differs from methods used in [8,13,20].

Keywords: Approximation theory; Minimal projection; Extension operators; Orlicz spaces

0. Introduction

Our aim is to find a minimal projection from a space of measurable functions on $[0,1]^n$ (equipped, e.g., with the L_p norm or the Orlicz norm) onto subspaces V of functions of "block independent variables" (the space V appears naturally in tensor product settings).

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Definition 0.1. Take any representation of n as a sum of k factors, i.e., $n = n_1 + n_2 + \cdots + n_k$. Then any function of the form (put $v_i = n_1 + \cdots + n_i$):

$$f(x_1, \dots, x_n) = f_1(x_1, \dots, x_{v_1}) + f_2(x_{v_1+1}, \dots, x_{v_2}) + \cdots + f_k(x_{v_{k-1}+1}, \dots, x_{v_k})$$

we will call a function of block independent variables corresponding to the given representation $n = n_1 + \cdots + n_k$.

This problem was solved in the case of L_p norm for $n = 2$ and the partition $2 = 1 + 1$ by Cheney, Halton and Light (see [8,13,20]). They gave a formula for a minimal projection in a discrete case too. Here we will present a new formula for minimal and co-minimal projections in more general settings. What is more, the techniques used here differ from which they use (we will combine the Rudin theorem with the Chalmers–Metcalf theorem). Additionally, we will observe that our projections are also co-minimal.

Let us now introduce some basics facts concerning projections and some crucial theorems.

Let $\mathscr{P}(X, V)$, denote the set of all continuous linear projections from X, onto V, i.e.,

$$\mathscr{P}(X, V) = \{P \in \mathscr{L}(X, V) \colon P/_V = Id_V\}.$$

A projection $P_0 \in \mathscr{P}(X, V)$, is called *minimal* if

$$||P_0|| = \lambda(V, X) = \inf\{||P|| \colon P \in \mathscr{P}(X, V)\}.$$

The constant $\lambda(V, X)$, is called the *relative projection constant*.

A projection $P_0 \in \mathscr{P}(X, V)$, is called *co-minimal* if

$$||Id_X - P_0|| = \inf\{||Id_X - P|| \colon P \in \mathscr{P}(X, V)\}.$$

Minimal and co-minimal projections are important for two main reasons. The first is the Lebesgue Inequality:

$$||x - Px|| \leqslant ||Id_X - P|| \cdot dist(x, V) \leqslant (1 + ||P||) \cdot dist(x, V).$$

The above inequality gives us a "good" linear approximation (the smaller the numbers $||P||$, $||Id_X - P||$ the better approximation) of elements from X by elements from V.

The second reason is connected with the Hahn–Banach theorem; having a minimal projection we can linearly extend any functional $v^* \in V^*$ to X^* (by setting $x^* = v^* \circ P$), or equivalently we can speak of a linear extension of the operator $Id_V : V \to V$ to X of the smallest possible norm.

For some surveys on minimal projections the reader is referred to [4,6,9,10,14,22].

To present the main results we will need the following notions.

Definition 0.2. For any $s_1, s_2, \dots, s_n \in [0, 1)$ consider the transformation $I_{s_1,\dots,s_n}$ which acts on any function f according to the formula

$$(I_{s_1,\dots,s_n} f)(a_1, \dots, a_n) = f(a_1 + s_1, \dots, a_n + s_n) \quad \text{for any } (a_1, \dots, a_n) \in [0, 1)^n.$$

The addition in the above formula is considered *modulo* $[0, 1]$. Therefore, such transformations form a group which is isomorphic to the n dimensional torus T_n, hence compact.

We will, later on, prove the following main theorem (see Section 2, Theorems 2.10 and 2.11).

Theorem 0.3. *Let φ fulfills $(\triangle_2)$ condition and consider the Orlicz space $X = L^{\varphi}([0,1]^n)$ equipped with the Orlicz or the Luxemburg norm (particularly the theorem will hold for L^p spaces for $p \in [1, +\infty)$). Fix any representation $n = n_1 + \cdots + n_k$ and let V be a subspace consisting of functions of block independent variables corresponding to the given representation of n (see Definition 0.1). Then the projection $Q : X \to V$, given on any $f \in L^{\varphi}([0,1]^n)$ by the formula*

$$\begin{aligned} Q(f) = & \int_{[0,1]^{n-n_1}} I_{0,u_2,\ldots,u_k}(f)\, du_2\, du_3 \ldots du_k \\ & + \int_{[0,1]^{n-n_2}} I_{u_1,0,u_3,\ldots,u_k}(f)\, du_1\, du_3 \ldots du_k \\ & + \cdots + \int_{[0,1]^{n-n_k}} I_{u_1,\ldots,u_{k-1},0}(f)\, du_1\, du_2 \ldots du_{k-1} \\ & - (k-1) \int_{[0,1]^n} I_{u_1,\ldots,u_k}(f)\, du_1 \ldots du_k \end{aligned}$$

(u_1 covers variables $x_1, \ldots, x_{n_1}$, u_2 variables $x_{n_1+1}, \ldots, x_{n_1+n_2}$ and so on, hence we have $(x_1, x_2, \ldots, x_n) = (u_1, u_2, \ldots, u_k)$) is both minimal and co-minimal in the set $\mathscr{P}(X, V)$.

The proof will use two main theorems which, for the sake of completion, we state below. We will also need some additional definitions.

Definition 0.4. Suppose that a Banach space X and a topological group G are related in the following manner: to every $s \in G$ corresponds a continuous linear operator $T_s : X \to X$ such that

$$T_e = I, \quad T_{st} = T_s T_t \quad (s \in G, t \in G).$$

Under these conditions, G is said to act as a group of linear operators on X.

Definition 0.5. A map $L : X \to X$ commutes with G if $T_g L T_{g^{-1}} = L$ for every $g \in G$.

Theorem 0.6 ((Rudin) Wojtaszczyk [27, III.B.13]). *Let X be a Banach space and V a complemented subspace, i.e., $\mathscr{P}(X, V) \neq \emptyset$. Let G be a compact group which acts as a group of linear operators on X such that*

1. *$T_g(x)$ is a continuous function of g, for every $x \in X$,*
2. *$T_g(V) \subset V$, for all $g \in G$.*
3. *T_g are isometries, for all $g \in G$.*

Assume furthermore there exists only one projection $P : X \to V$ which commutes with G. Then this projection is minimal and cominimal.

Fix any projection Q from X onto V, then the projection P (this unique one which commutes with G) is defined by

$$P(x) = \int_G T_g Q T_{g^{-1}}(x)\, dg \quad \textit{for } x \in X,$$

where dg denotes the normalized Haar measure on G.

This theorem, however, does not imply that this projection is the unique minimal projection as there could be projections which do not commute with G but still have a minimal norm. For applications of the above theorem and related results see, e.g., [7,8,11,13,15,16,19,20,22,27].

Below we assume that X is a normed space and V is a finite-dimensional subspace.

Definition 0.7. A pair $(x, y) \in S(X^{**}) \times S(X^*)$ will be called an *extremal pair* for $P \in \mathscr{P}(X, W)$ iff $y(P^{**}x) = ||P||$, where $P^{**} : X^{**} \to V$ is the second adjoint extension of P to X^{**} (S denotes here a unit sphere). Let $\mathscr{E}(P)$ be the set of all extremal pairs for P.

To each $(x, y) \in \mathscr{E}(P)$ we associate the rank-one operator $y \otimes x$ from X to X^{**} given by $(y \otimes x)(z) = y(z)x$ for $z \in X$.

Theorem 0.8 (Chalmers and Metcalf [4, Theorem 1]). *A projection $P \in \mathscr{P}(X, V)$ has a minimal norm if and only if the closed convex hull of $\{y \otimes x\}_{(x,y) \in \mathscr{E}(P)}$ contains an operator E_P for which V is an invariant subspace.*

The operator E_P is given by the formula

$$E_P = \int_{\mathscr{E}(P)} y \otimes x\, d\mu(x, y) : X \to X^{**},$$

where μ is a probability Borel measure on $\mathscr{E}(P)$.

For applications of the above theorem see, e.g., [1–4,17,23,26].

For some other methods used to find a minimal projections see, e.g., [6,9,12,14,18,24,25].

1. Preliminary results

We will prove our theorem in more general settings than those in Theorem 0.3. To proceed we will need some general notions first.

Fix a representation $n = n_1 + \cdots + n_k$.

For any integer m consider the following partition of the cube $[0,1)^n$ into m^n smaller cubes

$$K_{i_1,i_2,\dots,i_n} = \left[i_1, i_1 + \frac{1}{m}\right) \times \left[i_2, i_2 + \frac{1}{m}\right) \times \cdots \times \left[i_n, i_n + \frac{1}{m}\right), \tag{1.1}$$

where $i_1, \dots, i_n \in C = \{0, \frac{1}{m}, \dots, \frac{m-1}{m}\}$.

Definition 1.1. Let X_m be the linear space generated by characteristic functions of cubes $K_{i_1,i_2,\dots,i_n}$, and Z_l $(l = 1, \dots, k)$ be the linear space generated by characteristic functions of cubes $K_{j_1,j_2,\dots,j_{n_l}}$. Let V_m be the subspace of X_m of functions of block independent variables for which in their representation as a sum $f = f_1 + \cdots + f_k$ we have $f_l \in Z_l$ (see Definition 0.1).

Moreover, put

$$Y = \bigcup_{m=2^u,\, u\in\mathbb{N}} X_m \quad \text{and} \quad W = \bigcup_{m=2^u,\, u\in\mathbb{N}} V_m. \tag{1.2}$$

Now take any norm on Y, let X be the smallest Banach space containing Y and let V be the closure of W in the space X.

Consider transformations defined in Definition 0.2, they form a group with a natural action

$$I_{s_1,\dots,s_n} \circ I_{t_1,\dots,t_n} = I_{s_1+t_1,\dots,s_n+t_n} \quad \text{for any } s_1, \dots, s_n \text{ and } t_1, \dots, t_n, \tag{1.3}$$

where $s_i + t_i$ is considered as addition *modulo* $[0,1]$.

Definition 1.2. Consider a projection $L_m : Y \to X_m$ given, on any $f \in Y$, by the formula

$$L_m f = \sum_{i_1,\dots,i_n \in C} \left(m^n \int_{K_{i_1,\dots,i_n}} f(u_1, \dots, u_n)\, du_1 \dots du_n \right) \chi_{K_{i_1,\dots,i_n}}, \tag{1.4}$$

where $C = \{0, \frac{1}{m}, \dots, \frac{m-1}{m}\}$.

Condition 1.3. Assume the norm $\|\cdot\|$ on Y fulfills the following conditions:

(1) for any $s_1, \dots, s_n \in [0,1)$ the transformation $I_{s_1,\dots,s_n}$ (see Definition 0.2) is an isometry on X.
(2) for any $f \in X$ a function $(s_1, \dots, s_n) \to I_{s_1,\dots,s_n}(f)$ is continuous.
(3) There is a constant C such that $\|L_m\| \leqslant C$, for any integer m.

Points (1) and (2) enable us to apply the Rudin Theorem (Theorem 0.6) so we can exchange them with the condition: assumptions of Theorem 0.6 are fulfilled.

Consider the projection $Q: Y \to W$ given by the formula

$$\begin{aligned} Q(f) = & \int_{[0,1]^{n-n_1}} I_{0,u_2,\dots,u_k}(f)\, du_2\, du_3 \dots du_k \\ & + \int_{[0,1]^{n-n_2}} I_{u_1,0,u_3,\dots,u_k}(f)\, du_1\, du_3 \dots du_k \\ & + \cdots + \int_{[0,1]^{n-n_k}} I_{u_1,\dots,u_{k-1},0}(f)\, du_1\, du_2 \dots du_{k-1} \\ & - (k-1) \int_{[0,1]^n} I_{u_1,\dots,u_k}(f)\, du_1 \dots du_k \end{aligned} \tag{1.5}$$

(u_1 covers variables $x_1, \dots, x_{n_1}$, u_2 variables $x_{n_1+1}, \dots, x_{n_1+n_2}$ and so on, hence we have $(x_1, x_2, \dots, x_n) = (u_1, u_2, \dots, u_k)$).

Our aim is to prove that $\tilde{Q}: X \to V$, the unique extension of the above projection $Q: Y \to W$, is minimal in $\mathscr{P}(X, V)$.

Remark 1.4. Any operator $T \in \mathscr{L}(Y, Y)$ can be uniquely extended to the operator $\tilde{T} \in \mathscr{L}(X, X)$ by

$$\tilde{T}(f) := \lim_{k \to \infty} T(f_k),$$

where $\{f_k\}$ is any given sequence in Y converging to f. This extension is obviously norm-preserving ($||\tilde{T}|| = ||T||$).

Theorem 1.5. *Assume that the norm on Y fulfills Condition* 1.3. *For any m a projection* $\widetilde{L_m}: X \to X_m$ (*the extension of* $L_m: Y \to X_m$ (1.4) *from the above theorem*) *has the properties*

(1) $\widetilde{L_m}(V) \subset V_m$;
(2) *there is a constant* $C > 0$ *such that* $||\widetilde{L_m}|| \leqslant C$, *for any integer m*;
(3) *for any* $f \in X$, *letting* $m \to \infty$ *we have* $\widetilde{L_m} f \to f$.

Proof. X_m, V_m are closed and we have $L_m(W) \subset V_m$, which gives (1). The proof of (2) is straightforward. To prove (3) observe that if $f \in X_{m_0}$ then $L_m(f) = f$, for any $m \geqslant m_0$. Therefore, the pointwise convergence $\widetilde{L_m} f \to f$ holds for any $f \in Y$. And Y is a dense set in X and the norms of operators $\widetilde{L_m}$ are bounded by C. Applying now the Banach-Steinhaus theorem we get $\widetilde{L_m} f \to f$ for any $f \in X$. $\square$

Theorem 1.6. *Let $Q_m := Q/_{X_m}$. Then Q_m is a projection from X_m onto V_m and we have a formula*

$$Q_m(f) = \frac{1}{m^{n-n_1}} \sum_{s_2,\dots,s_k} I_{0,s_2,\dots,s_k}(f) + \frac{1}{m^{n-n_2}} \sum_{s_1,s_3,\dots,s_k} I_{s_1,0,s_3,\dots,s_k}(f) + \cdots + \frac{1}{m^{n-n_k}} \sum_{s_1,\dots,s_{k-1}} I_{s_1,\dots,s_{k-1},0}(f) - \frac{k-1}{m^n} \sum_{s_1,\dots,s_k} I_{s_1,\dots,s_k}(f). \tag{1.6}$$

Note that s_l are "block integers" corresponding to the representation of n. Here $\sum_{s_1,\dots,s_k}$ means the sum is over all possible choices of $s_1,\dots,s_k$, i.e., over all possible n_l-tuples s_l with each coordinates in a set $C = \{0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}\}$. We will use this notion in many places later on.

Proof. Take $\{\chi_{K_{i_1,\dots,i_n}}\}$, the basis of X_m. Using the identity

$$\chi_{K_{i_1,\dots,i_n}}(x_1,\dots,x_n) = \chi_{K_{i_1}}(x_1)\chi_{K_{i_2}}(x_2)\cdots\chi_{K_{i_n}}(x_n),$$

where $\chi_{K_{i_l}} = \chi_{[\frac{i_1}{m},\frac{i_1+1}{m})}$ and the Fubini Theorem (put $v_i = n_1 + \cdots + n_i$, $j_i = i_{v_{i-1}},\dots,i_{v_i}$ and $y_i = (x_{v_{i-1}+1},\dots,x_{v_i})$ is a "block variable"), we obtain

$$\begin{aligned}
&\left(\int_{[0,1]^{n-n_1}} I_{0,u_2,\dots,u_k}(\chi_{K_{i_1,\dots,i_n}})\, du_2\dots du_k\right)(x_1,\dots,x_n) \\
&= \left(\int_{[0,1]^{n-n_1}} I_{0,u_2,\dots,u_k}(\chi_{K_{i_1,\dots,i_n}})\, du_2\dots du_k\right)(y_1,\dots,y_k) \\
&= \int_{[0,1]^{n-n_1}} \chi_{K_{j_1}}(y_1)\cdots\chi_{K_{j_2}}(y_2+u_2)\cdots\chi_{K_{j_k}}(y_k+u_k)\, du_2\dots du_k \\
&= \chi_{K_{j_1}}(y_1)\left(\int_{[0,1]^{n_2}} \chi_{K_{j_2}}(y_2+u_2)\, du_2\right)\cdots\left(\int_{[0,1]^{n_k}} \chi_{K_{j_k}}(y_k+u_k)\, du_k\right) \\
&= \chi_{K_{j_1}}(y_1)\left(\int_{[0,1]^{n_2}} \chi_{K_{j_2}}(u_2)\, du_2\right)\cdots\left(\int_{[0,1]^{n_k}} \chi_{K_{j_k}}(u_k)\, du_k\right) \\
&= \chi_{K_{j_1}}(y_1)\cdot\frac{1}{m^{n_2}}\cdots\frac{1}{m^{n_k}} \\
&= \frac{1}{m^{n-n_1}}\cdot\chi_{K_{j_1}}(y_1) = \frac{1}{m^{n-n_1}}\cdot\chi_{K_{u_1}}(x_1)\cdots\chi_{K_{u_{v_1}}}(x_{v_1}).
\end{aligned}$$

Reasoning in the same way for $I_{u_1,\dots,u_{l-1},0,u_{l+1},\dots,u_k}$ $(l = 0,1,\dots,k)$ we arrive at

$$\left(\int_{[0,1]^{n-n_l}} I_{u_1,\dots,u_{l-1},0,u_{l+1},\dots,u_k}(\chi_{K_{i_1,\dots,i_n}})\, du_1\dots d_{l-1}d_{l+1}\dots du_k\right) = \frac{1}{m^{n-n_l}}\cdot\chi_{K_{j_l}}, \tag{1.7}$$

for any $l \in \{0, 1, \dots, k\}$. In the same way we obtain

$$\left(\int_{[0,1]^{n-n_l}} I_{u_1,\dots,u_k}(\chi_{K_{i_1,\dots,i_n}})\, du_1 \dots du_k \right) = \frac{1}{m^n}. \tag{1.8}$$

Combining these two above we get

$$Q(\chi_{K_{i_1,\dots,i_n}}) = \frac{1}{m^{n-n_1}} \cdot \chi_{K_{j_1}} + \cdots + \frac{1}{m^{n-n_k}} \cdot \chi_{K_{j_k}} - \frac{k-1}{m^n}. \tag{1.9}$$

Now observe that

$$\begin{aligned}\frac{1}{m^{n-n_1}} \sum_{s_2,\dots,s_k} I_{0,s_2,\dots,s_k}(\chi_{K_{i_1,\dots,i_n}}) &= \frac{1}{m^{n-n_1}} \sum_{s_2,\dots,s_k} I_{0,s_2,\dots,s_k}(\chi_{K_{j_1,\dots,j_k}}) \\ &= \frac{1}{m^{n-n_1}} \sum_{s_2,\dots,s_k} \chi_{K_{j_1,\, j_2+s_2,\dots,\, j_k+s_k}} \\ &= \frac{1}{m^{n-n_1}} \sum_{s_2,\dots,s_k} \chi_{K_{j_1,s_2,\dots,s_k}} \\ &= \frac{1}{m^{n-n_1}} \chi_{K_{j_1}}.\end{aligned}$$

In the same way we can prove the equalities

$$\begin{aligned}&\frac{1}{m^{n-n_1}} \sum_{s_1,\dots,s_{l-1},s_l,\dots,s_k} I_{s_1,\dots,s_{l-1},0,s_l,\dots,s_k}(\chi_{K_{i_1,\dots,i_n}}) \\ &\quad = \frac{1}{m^{n-n_1}} \sum_{s_1,\dots,s_{l-1},s_l,\dots,s_k} I_{s_1,\dots,s_{l-1},0,s_l,\dots,s_k}(\chi_{K_{j_1,\dots,j_k}}) \\ &\quad = \frac{1}{m^{n-n_l}} \chi_{K_{j_l}}.\end{aligned}$$

Therefore, two linear operators (given by (1.5) and (1.6)) coincides on a basis of X_m and consequently are equal on X_m. $\square$

Theorem 1.7. *Assume the norm on Y fulfills Condition* 1.3 (*only point* (1) *is needed*). *Then* $||Q|| \leqslant 2k-1$, *where* $Q: Y \to W$ *is given by* (1.5), *and hence for its extension* $\tilde{Q}: X \to V$ *we have also* $||\tilde{Q}|| \leqslant 2k-1$. *In particular* $\mathscr{P}(X, V) \neq \emptyset$.

Proof. Since $I_{s_1,\dots,s_k}$ are isometries then

$$\begin{aligned}||Q_m(f)|| = \Bigg\| &\frac{1}{m^{n-n_1}} \sum_{s_2,\dots,s_k} I_{0,s_2,\dots,s_k}(f) + \frac{1}{m^{n-n_2}} \sum_{s_1,s_3,\dots,s_k} I_{s_1,0,s_3,\dots,s_k}(f) \\ &+ \cdots + \frac{1}{m^{n-n_k}} \sum_{s_1,\dots,s_{k-1}} I_{s_1,\dots,s_{k-1},0}(f) - \frac{k-1}{m^n} \sum_{s_1,\dots,s_k} I_{s_1,\dots,s_k}(f) \Bigg\|\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{1}{m^{n-n_1}} \sum_{s_2,\dots,s_k} ||I_{0,s_2,\dots,s_k}(f)|| + \frac{1}{m^{n-n_2}} \sum_{s_1,s_3,\dots,s_k} ||I_{s_1,0,s_3,\dots,s_k}(f)|| \\
&\quad + \cdots + \frac{1}{m^{n-n_k}} \sum_{s_1,\dots,s_{k-1}} ||I_{s_1,\dots,s_{k-1},0}(f)|| + \frac{k-1}{m^n} \sum_{s_1,\dots,s_k} ||I_{s_1,\dots,s_k}(f)|| \\
&= \frac{1}{m^{n-n_1}} \sum_{s_2,\dots,s_k} ||f|| + \frac{1}{m^{n-n_2}} \sum_{s_1,s_3,\dots,s_k} ||f|| \\
&\quad + \cdots + \frac{1}{m^{n-n_k}} \sum_{s_1,\dots,s_{k-1}} ||f|| + \frac{k-1}{m^n} \sum_{s_1,\dots,s_k} ||f|| \\
&= (2k+1)||f||.
\end{aligned}$$

Hence $||Q/_{X_m}|| \leqslant 2k-1$ and consequently by the definition of Y (see (1.2)) $||Q/_Y|| \leqslant 2k-1$. But Y is dense in X so by Remark 1.4 $||\tilde{Q}|| \leqslant 2k-1$. □

2. Main results

In this section every linear operator belonging to $\mathscr{L}(Y, Y)$ and its extension to $\mathscr{L}(X, X)$ we will, for simplicity, denote by the same letter.

We will also assume that the norm on Y fulfills Condition 1.3 and that we are given the representation $n = n_1 + \cdots + n_k$.

The following theorem is crucial for our reasoning, the method used here seems to be useful for proving that some particular projection is the only one which commutes with a given group. What is more interesting the Chalmers–Metcalf theorem will come in handy.

Theorem 2.1. *There is only one projection $S \in \mathscr{P}(X_m, V_m)$ which commutes with isometries $I_{s_1,\dots,s_k}$, where s_i are block variables (i.e., s_i is a n_i tuple) with each coordinate in a set $C = \{0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}\}$. Moreover, this projection is given by the formula* (1.6) *in Theorem* 1.6.

Proof. Consider the following elements of V_m:

$$\begin{aligned}
&\chi_{K_{i_1,0,\dots,0}} \quad i_1 \text{ is a } n_1 \text{ tuple with coordinates in } C \\
&\chi_{K_{0,i_2,0,\dots,0}} \quad i_2 \text{ is a } n_2 \text{ tuple with coordinates in } C \\
&\vdots \\
&\chi_{K_{0,\dots,0,i_k}} \quad i_k \text{ is a } n_k \text{ tuple with coordinates in } C. \qquad (2.1)
\end{aligned}$$

These elements span V_m, though they do not form the basis, since *dim* $V_m = m^{n_1} + \cdots + m^{n_k} - k + 1$.

The elements $\chi_{K_{i_1,i_2,\dots,i_k}}$ form a basis of X_m and in this basis any projection $S\in\mathscr{P}(X_m,V_m)$ can be written as

$$S(\chi_{K_{i_1,i_2,\dots,i_k}})=\sum_{j_1,j_2,\dots,j_k} a_{i_1,i_2,\dots,i_k}^{j_1,j_2,\dots,j_k}\chi_{K_{j_1,j_2,\dots,j_k}}.$$

Hence, our theorem transforms to the statement that there is only one solution to the following system of linear equations ($a_{i_1,i_2,\dots,i_k}^{j_1,j_2,\dots,j_k}$ are treated as variables)

$$\begin{aligned}&(1)\ I_{i_1,i_2,\dots,i_k}\circ S=S\circ I_{i_1,i_2,\dots,i_k},\quad \text{for any } n\text{-tuple with coordinates in } C;\\&(2)\ S(\chi_{K_{i_1,i_2,\dots,i_k}})\in V_m,\quad \text{for any } n\text{-tuple with coordinates in } C;\\&(3)\ S(\chi_{K_{i_1,0,\dots,0}})=\chi_{K_{i_1,0,\dots,0}},\dots,S(\chi_{K_{0,\dots,0,i_k}})=\chi_{K_{0,\dots,0,i_k}}.\end{aligned}\tag{2.2}$$

It is easy to see that these equations are linear. (2) and (3) state that S is a projection from X_m onto V_m (point (2)) states that some of $a_{i_1,i_2,\dots,i_k}^{j_1,j_2,\dots,j_k}$ are zeros, since we can choose from $\chi_{K_{i_1,i_2,\dots,i_k}}$ the basis of V_m, and point (3) gives $S/_{V_m}=id_{V_m}$, since the elements (2.1) span V_m).

Observe that system (2.2) has a solution—a projection (1.6). Indeed from the form of this projection (1.6) and, if we realize that (1.3) gives that transformations $I_{i_1,i_2,\dots,i_k}$ commute with each other, we get (1), moreover (2) and (3) are automatically fulfilled because the projection (1.6) is a projection from X_m onto V_m.

Since X_m is finite dimensional, and in finite-dimensional spaces all linear operators are continuous (in any norm) then the solution of system (2.2) does not depend on any particular norm. Therefore we choose a Hilbert norm on X_m (the norm which makes X_m a Hilbert space).

Take any projection S which commutes with the group of isometries $I_{s_1,\dots,s_k}$ (using the Chalmers–Metcalf theorem we will prove that this projection has to be minimal and since in Hilbert spaces there is only one minimal projection onto any closed subspace (the orthogonal projection) then all these projections have to be equal to orthogonal projection; hence there is only one solution to the (2.2)).

Take any extremal pair $(y,x)\in\mathscr{E}(S)$ (i.e., $|y(S(x))|=||S||$, recall that we are considering the Hilbert norm). Observe that also $(I_{i_1,\dots,i_k}(y),I_{i_1,\dots,i_k}(x))\in\mathscr{E}(S)$. Indeed, since S commutes with $I_{i_1,\dots,i_k}$ and $y(I^{-1}_{i_1,\dots,i_k}(z))=(I_{i_1,\dots,i_k}y)(z)$, we have

$$\begin{aligned}||S||=|y(S(x))|&=|y((I_{i_1,\dots,i_k})^{-1}SI_{i_1,\dots,i_k}(x))|\\&=|y((I_{i_1,\dots,i_k})^{-1}(SI_{i_1,\dots,i_k}(x)))|=|(I_{i_1,\dots,i_k}y)(S(I_{i_1,\dots,i_k}(x)))|.\end{aligned}$$

Thus, we can define the operator E_S (similarly to [26]) by

$$E_S=\frac{1}{m^n}\sum_{i_1,\dots,i_k}(I_{i_1,\dots,i_k}y)\otimes(I_{i_1,\dots,i_k}x):X_m\to X_m.\tag{2.3}$$

Now, we will prove that $E_S(V_m)\subset V_m$, which will gives us that E_S is a Chalmers–Metcalf operator. To do so, take any $\chi_{K_{s_1,0,\dots,0}}$; using $y(I^{-1}_{i_1,\dots,i_k}(z))=(I_{i_1,\dots,i_k}y)(z)$ we compute

$$\begin{aligned} m^n E_S(\chi_{K_{s_1,0,\dots,0}}) &= \sum_{i_1,\dots,i_k} (I_{i_1,\dots,i_k} y) \otimes (I_{i_1,\dots,i_k} x)(\chi_{K_{s_1,0,\dots,0}}) \\ &= \sum_{i_1,\dots,i_k} (I_{i_1,\dots,i_k} y)(\chi_{K_{s_1,0,\dots,0}})\, I_{i_1,\dots,i_k}(x) \\ &= \sum_{i_1,\dots,i_k} y(I^{-1}_{i_1,\dots,i_k}(\chi_{K_{s_1,0,\dots,0}}))\, I_{i_1,\dots,i_k}(x) \\ &= \sum_{i_1,\dots,i_k} y(I_{-i_1,\dots,-i_k})(\chi_{K_{s_1,0,\dots,0}}))\, I_{i_1,\dots,i_k}(x) \\ &= \sum_{i_1{}',\dots,i_k{}'} y(I_{i_1{}',\dots,i_k{}'}(\chi_{K_{s_1,0,\dots,0}}))\, I_{-i_1{}',\dots,-i_k{}'}(x); \end{aligned}$$

in the last equality we have changed the summation putting $i_l{}' = -i_l$ modulo $[0, 1)$.

Observe that $I_{i_1,\dots,i_k}(\chi_{K_{s_1,0,\dots,0}}) = \chi_{K_{s_1+i_1,0,\dots,0}}$, so using the previous computation we have

$$\begin{aligned} m^n E_S(\chi_{K_{s_1,0,\dots,0}}) &= \sum_{i_1,\dots,i_k} y(I_{i_1,\dots,i_k}(\chi_{K_{s_1,0,\dots,0}}))\, I_{-i_1,\dots,-i_k}(x) \\ &= \sum_{i_1,\dots,i_k} y(\chi_{K_{s_1+i_1,0,\dots,0}})\, I_{-i,-j}(x) \\ &= \sum_{i_1} y(\chi_{K_{s_1+i_1,0,\dots,0}}) \left(\sum_{i_2,\dots,i_k} I_{-i_1,\dots,-i_k}(x) \right). \end{aligned} \tag{2.4}$$

Consider now the term in brackets in the last equality. Observe that for any $t_1, \dots, t_k$ (x_i are block variables with each coordinate in a set $C = \{0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}\}$) we get

$$\begin{aligned} \left(\sum_{i_2,\dots,i_k} I_{-i_1,\dots,-i_k}(x) \right)(t_1, \dots, t_k) &= \sum_{i_2,\dots,i_k} x(t_1 - i_1, t_2 - i_2, \dots, t_k - i_k) \\ &= \sum_{i_2{}',\dots,i_k{}'} x(t_1 - i_1, i_2{}', \dots, i_k{}'). \end{aligned}$$

And since $x \in X_m$ (therefore the values of x at k-tuples $t_1, \dots, t_k$ determine x), then from the above, the last sum depends only on the first n_1 variables (t_1th is a n_1 tuple), hence it belongs to V_m, i.e., we have

$$\left(\sum_{i_2,\dots,i_k} I_{-i_1,\dots,-i_k}(x) \right) \in V_m \quad \text{for any } i_2, \dots, i_k.$$

This, with (2.4), gives $E_S(\chi_{K_{s_1,0,\dots,0}}) \in V_m$ (for any s_1). Now we can conduct the same reasoning for other elements from (2.1). But since elements from (2.1) span X_m we get $E_S(X_m) \subset V_m$.

Therefore, indeed, E_S is a Chalmers–Metcalf operator and from Theorem 0.8 a projection S is minimal (We have proved: If a projection S commutes with isometries $I_{i_1,\dots,i_k}$ then it has to be a minimal projection in the Hilbert norm). But we know that in Hilbert spaces a minimal projection on any closed subspace is unique (it is the orthogonal projection) and the projection Q_m given by (1.6) is minimal (since it commutes with $I_{i_1,\dots,i_k}$). Hence $S = Q_m$ and as a consequence the system (2.2) has only one solution. □

Observe that, if we use the above theorem and apply the Rudin theorem (Theorem 0.6) we will get the minimality of the projection (1.6) in a discrete case (we can then think of X_m as a m-dimensional matrix space), i.e., we get the minimality of projection (1.6) in $\mathscr{P}(X_m, V_m)$. Hence we have

Theorem 2.2. *Consider X_m with the norm fulfilling the following condition*:

$$\text{for any } s_1,\dots,s_k \text{ transformations } I_{s_1,\dots,s_k} \text{ are isometries } \Bigg(s_i \text{ are block variables, i.e., } s_i \text{ is a } n_i \text{ tuple with each coordinate in a set } C = \left\{0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}\right\}\Bigg). \tag{2.5}$$

Then the projection $Q_m : X_m \to V_m$ given by the formula

$$Q_m(f) = \frac{1}{m^{n-n_1}} \sum_{s_2,\dots,s_k} I_{0,s_2,\dots,s_k}(f) + \frac{1}{m^{n-n_2}} \sum_{s_1,s_3,\dots,s_k} I_{s_1,0,s_3,\dots,s_k}(f) + \cdots + \frac{1}{m^{n-n_k}} \sum_{s_1,\dots,s_{k-1}} I_{s_1,\dots,s_{k-1},0}(f) - \frac{k-1}{m^n} \sum_{s_1,\dots,s_k} I_{s_1,\dots,s_k}(f) \tag{2.6}$$

is both minimal and co-minimal in the set $\mathscr{P}(X_m, V_m)$.

The above theorem is a generalization of results of Cheney and Light [8], where they have found a minimal projection for $n = 2$ (and the partition $2 = 1 + 1$). Also, it is proved in [26] that for $n = 2$ there is only one minimal projection in a set $\mathscr{P}(X_m, V_m)$ (we have the uniqueness of a minimal projection). Following the same approach we can also prove uniqueness for any n in a discrete case (but this is out of the scope of this paper so we do not provide any details here).

Observe, now that if some norm does not fulfill condition (2.5) (as for instance in the case of Musielak–Orlicz sequence spaces) then we can slightly modify it as follows.

Remark 2.3. Consider X_m with any given norm $||\cdot||_0$. Then the norm $||\cdot||$ given on any element $x \in X_m$ by

$$||x|| := \frac{1}{m^n} \sum_{s_1,\dots,s_k} ||I_{s_1,\dots,s_k} x||_0$$

fulfills condition (2.5) from Theorem 2.2.

Lemma 2.6. *Fix m. Then $I_{s_1,\dots,s_k} \circ L_m = L_m \circ I_{s_1,\dots,s_k}$, for any $s_1,\dots,s_k$.*

Proof. Let $(s_1, \dots, s_k) = (i_1, \dots, i_n)$. Changing variables $(v_l = u_l + i_l)$ we get

$$\begin{aligned} L_m(I_{i_1,\dots,i_n} f) &= \sum_{j_1,\dots,j_n} \left(m^n \int_{K_{j_1,\dots,j_n}} f(u_1 + i_1, \dots, u_n + i_n)\, du_1 \dots du_n \right) \chi_{K_{j_1,\dots,j_n}} \\ &= \sum_{j_1,\dots,j_n} \left(m^n \int_{K_{j_1 - i_1,\dots,j_n - i_n}} f(v_1, \dots, v_n)\, dv_1 \dots dv_n \right) \chi_{K_{j_1,\dots,j_n}} \\ &= \sum_{j_1',\dots,j_n'} \left(m^n \int_{K_{j_1',\dots,j_n'}} f(v_1, \dots, v_n)\, dv_1 \dots dv_n \right) \chi_{K_{j_1'+i_1,\dots,j_n'+i_n}} \\ &= (I_{i_1,\dots,i_n})(L_m f), \end{aligned}$$

in the third equality we have changed summation putting $j_l' = j_l - i_l$ modulo $[0, 1)$. □

Now we will prove the crucial theorem.

Theorem 2.7. *There is only one projection* $P \in \mathscr{P}(X, V)$ *which commutes with the group* T_n (*see Definition* 0.2). *This projection is given by formula* (1.5).

Proof. Take any projection P which commutes with T_n. Put

$$S_m = L_m \circ P : X \to V_m \tag{2.7}$$

and

$$\widetilde{S_m} = S_m/_{X_m} : X_m \to V_m. \tag{2.8}$$

From the properties of the projection L_m (described in Theorem 1.5) operator $\widetilde{S_m}$ is a linear projection from X_m onto V_m.

Let Q_m be the projection given by (1.6).

Fix any $i_1, \dots, i_n$. Since P commutes with T_n and using the above Lemma 2.6 we get

$$\begin{aligned} I_{i_1,\dots,i_n} \circ S_m &= I_{i_1,\dots,i_n} \circ L_m \circ P = (I_{i_1,\dots,i_n} \circ L_m) \circ P \\ &= (L_m \circ I_{i_1,\dots,i_n}) \circ P = L_m \circ (I_{i_1,\dots,i_n} \circ P) \\ &= L_m \circ (P \circ I_{i_1,\dots,i_n}) = (L_m \circ P) \circ I_{i_1,\dots,i_n} = S_m \circ I_{i_1,\dots,i_n}. \end{aligned}$$

And since $I_{i_1,\dots,i_n}(X_m) = X_m$, for any $i_1, \dots, i_n \in C = \{0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}\}$ we have also

$$I_{i_1,\dots,i_n} \circ \widetilde{S_m} = \widetilde{S_m} \circ I_{i_1,\dots,i_n} \quad \text{for any } i_1, \dots, i_n \in C. \tag{2.9}$$

From (2.9) and Theorem 2.1 the projections $\widetilde{S_m}$ have to be given by formula (1.6), therefore

$$L_m \circ P/_{X_m} = \widetilde{S_m} = Q_m \quad \text{for any } m \in \mathbb{N}. \tag{2.10}$$

Notice that projection Q given by (1.5) has property (see Theorem 1.6)

$$Q/_{X_m} = Q_m \quad \text{for any } m \in \mathbb{N}. \tag{2.11}$$

Fix any $f \in X$. From the definition of X (see Definition 1.1) we can choose a sequence $\{f_m\}$ such that $f_m \in X_m$ and $||f - f_m|| \to 0$ (with $m \to \infty$). From (2.10) and (2.11) and points (2), (3) from Theorem 1.5 we get

$$\begin{aligned} ||(P-Q)(f)|| &= ||(P - L_m \circ P)(f) + (L_m \circ P - Q)(f)|| \\ &\leqslant ||(P - L_m \circ P)(f)|| + ||(L_m \circ P - Q)(f)|| \\ &= ||(P - L_m \circ P)(f)|| + ||(L_m \circ P - Q)(f - f_m)|| \\ &\leqslant ||(P - L_m \circ P)(f)|| + (C\,||P|| + ||Q||) \cdot ||(f - f_m)||. \end{aligned}$$

And since letting $m \to \infty$ the right side of the above inequality tends to 0 we have $P = Q$. □

Combining the above Theorems 2.7 and 1.7 with the Rudin theorem (Theorem 0.6) and using the fact that Id_X commutes with the group T_n we get

Theorem 2.8. *Fix any partition* $n = n_1 + \cdots + n_k$. *Consider the space* Y (*see Definition* 1.1) *with a norm fulfilling Condition* 1.3. *Let* X *be the smallest Banach space containing* Y *and* V *be the closure of* W *in the space* X. *Then the projection* $\tilde{Q} : X \to V$, *which is the unique extension of projection* $Q : Y \to W$ *given by*

$$\begin{aligned} Q(f) = &\int_{[0,1]^{n-n_1}} I_{0,u_2,\ldots,u_k}(f)\,du_2\,du_3 \ldots du_k \\ &+ \int_{[0,1]^{n-n_2}} I_{u_1,0,u_3,\ldots,u_k}(f)\,du_1\,du_3 \ldots du_k \\ &+ \cdots + \int_{[0,1]^{n-n_k}} I_{u_1,\ldots,u_{k-1},0}(f)\,du_1\,du_2 \ldots du_{k-1} \\ &- (k-1) \int_{[0,1]^n} I_{u_1,\ldots,u_k}(f)\,du_1 \ldots du_k \end{aligned}$$

is both minimal and co-minimal in the set $\mathscr{P}(X, V)$.

Now, we will provide some examples of norms fulfilling Condition 1.3, hence we get the minimality of Q in many natural spaces.

We will start with L_p norms. Condition 1.3 is then fulfilled and since for $p \in [1, \infty)$ the closure of Y is the space $X = L^p([0,1]^n)$ we may state

Theorem 2.9. *Take $p \in [1, \infty)$ and fix any representation $n = n_1 + \cdots + n_k$. Then the projection Q given by the formula*

$$\begin{aligned} Q(f) = & \int_{[0,1]^{n-n_1}} I_{0,u_2,\ldots,u_k}(f)\, du_2\, du_3 \ldots du_k \\ & + \int_{[0,1]^{n-n_2}} I_{u_1,0,u_3,\ldots,u_k}(f)\, du_1\, du_3 \ldots du_k \\ & + \cdots + \int_{[0,1]^{n-n_k}} I_{u_1,\ldots,u_{k-1},0}(f)\, du_1\, du_2 \ldots du_{k-1} \\ & - (k-1) \int_{[0,1]^n} I_{u_1,\ldots,u_k}(f)\, du_1 \ldots du_k \end{aligned}$$

is both minimal and co-minimal from $X = L^p([0,1]^n)$ onto $V = L^p([0,1]^{n_1}) + \cdots + L^p([0,1]^{n_k})$ (i.e., V is a subspace consisting of functions of block independent variables corresponding to the given representation of n—see Definition 0.1).

Now we present further examples; we will need a notion of Orlicz spaces.

Let M be the set of all measurable functions with respect to the Lebesgue measure and finite almost everywhere, divided by the relation of equality almost everywhere.

Let ϕ be a convex function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0$ and $\phi \neq 0$.

The Orlicz space we will call the following space:

$$L^{\phi}([0,1]^n) := \left\{ f \in M \colon \lim_{\lambda \to 0^+} \left(\int_{[0,1]^n} \phi(\lambda\, |f(x_1, \ldots, x_n)|)\, dx_1 \ldots dx_n \right) \to 0 \right\}. \tag{2.12}$$

The space of finite elements of the Orlicz space is a space

$$E^{\phi}([0,1]^n) := \left\{ f \in M \colon \int_{[0,1]^n} \phi(\lambda\, |f(x_1, \ldots, x_n)|)\, dx_1 \ldots dx_n < +\infty, \text{ for any } \lambda > 0 \right\}. \tag{2.13}$$

The Orlicz modular corresponding to a given function ϕ is given on any function $f \in M$ by the formula

$$\rho_\phi(f) = \int_{[0,1]^n} \phi(|f(x_1, \ldots, x_n)|)\, dx_1 \ldots dx_n.$$

We can equip space L^ϕ (so also E^ϕ, since $E^\phi \subset L^\phi$) with the Luxemburg (2.14) or the Orlicz norm (2.15) (in case of the Orlicz norm we use the Amemiya formula)

$$||f||_\phi = \inf\{d > 0 \colon \rho_\phi(f/d) \leqslant 1\}, \tag{2.14}$$

$$|||f|||_\phi = \inf_{d>0} \{d + d\, \rho_\phi(f/d)\}. \tag{2.15}$$

The reader particularly interested in Orlicz spaces and its applications is referred to [5,21].

We can easily see that both these norms fulfill Condition 1.3 (since $\rho_\phi(I_{s,t}f) = \rho_\phi(f)$ and using the Jensen inequality we get $\rho_\phi(L_n f) \leqslant \rho_\phi(f)$). Since the closure of Y (see Definition 1.1) in both the Luxemburg and the Orlicz norm equals E^ϕ, using Theorem 2.8, we may state

Theorem 2.10. *Fix any representation $n = n_1 + \cdots + n_k$. Consider the space $E^\phi([0,1]^n)$ equipped with the Luxemburg or the Orlicz norm. Then the projection Q given by the formula*

$$\begin{aligned} Q(f) = & \int_{[0,1]^{n-n_1}} I_{0,u_2,\dots,u_k}(f)\, du_2\, du_3 \dots du_k \\ & + \int_{[0,1]^{n-n_2}} I_{u_1,0,u_3,\dots,u_k}(f)\, du_1\, du_3 \dots du_k \\ & + \cdots + \int_{[0,1]^{n-n_k}} I_{u_1,\dots,u_{k-1},0}(f)\, du_1\, du_2 \dots du_{k-1} \\ & - (k-1) \int_{[0,1]^n} I_{u_1,\dots,u_k}(f)\, du_1 \dots du_k \end{aligned}$$

is both minimal and co-minimal from $X = E^\phi([0,1]^n)$ onto $V = E^\phi([0,1]^{n_1}) + \cdots + E^\phi([0,1]^{n_k})$ (i.e., V is a subspace consisting of functions of block independent variables corresponding to the given representation of n—see Definition 0.1).

For the equality of spaces $L^\phi = E^\phi$ it is necessary and sufficient that $\phi \in (\triangle_2)$. We say $\phi \in (\triangle_2)$ if there is a constant $C > 0$ and $u_0 \geqslant 0$ such that

$$\phi(2u) \leqslant C\phi(u) \quad \text{for any } u \geqslant u_0. \tag{2.16}$$

Now we can reformulate Theorem 2.10 as follows.

Theorem 2.11. *Fix any representation $n = n_1 + \cdots + n_k$. Assume that the Orlicz function ϕ fulfills the condition $(\triangle_2)$ and consider the space $L^\phi([0,1]^n)$ equipped with the Luxemburg or the Orlicz norm. Then the projection Q given by the formula*

$$\begin{aligned} Q(f) = & \int_{[0,1]^{n-n_1}} I_{0,u_2,\dots,u_k}(f)\, du_2\, du_3 \dots du_k \\ & + \int_{[0,1]^{n-n_2}} I_{u_1,0,u_3,\dots,u_k}(f)\, du_1\, du_3 \dots du_k \\ & + \cdots + \int_{[0,1]^{n-n_k}} I_{u_1,\dots,u_{k-1},0}(f)\, du_1\, du_2 \dots du_{k-1} \\ & - (k-1) \int_{[0,1]^n} I_{u_1,\dots,u_k}(f)\, du_1 \dots du_k \end{aligned}$$

is both minimal and co-minimal from $X = L^{\phi}([0,1]^n)$ *onto* $V = L^{\phi}([0,1]^{n_1}) + \dots + L^{\phi}([0,1]^{n_k})$ (*i.e.*, *V is a subspace consisted of functions of block independent variables corresponding to the given representation of n—see Definition* 0.1).

Modifying norms fulfilling Condition 1.3 we can obtain further examples fulfilling this condition as follows.

Theorem 2.12. *Assume that norms* $\|\cdot\|_1, \dots, \|\cdot\|_k$ *fulfill Condition* 1.3. *Then also the norms*

(1) $\|\cdot\| := \max\{\|\cdot\|_1, \dots, \|\cdot\|_k\}$,
(2) $\|\cdot\| := ((\|\cdot\|_1)^p + \dots + (\|\cdot\|_k)^p)^{1/p}$, *for* $p \in [1, \infty)$,

fulfill Condition 1.3.

Theorem 2.13. *Take functions* $\phi_1, \dots, \phi_k$ *as in definition of the Orlicz space. Then the Luxemburg and the Orlicz norm generated by the following modulars*:

(1) $\rho(\cdot) := \max\{\rho_{\phi_1}(\cdot), \dots, \rho_{\phi_k}(\cdot)\}$,
(2) $\rho(\cdot) := (\alpha_1(\rho_{\phi_1}(\cdot))^p + \dots + \alpha_k(\rho_{\phi_k}(\cdot))^p)^{1/p}$, *for any* $\alpha_1, \dots, \alpha_k > 0$.

fulfill Condition 1.3.

Proof. We get it easily if we first observe that $\rho_{\phi_i}(I_{s,t}f) = \rho_{\phi_i}(f)$ and $\rho_{\phi_i}(L_n f) \leqslant \rho_{\phi_i}(f)$. $\square$

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